International Journal of Theoretical Physics, Vol. 8, No. 4 (1973), pp. 237–242

Finiteness of Relative Acceleration at the Surface $r = 2m$

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Received: 4 *December* 1972

Abstract

We have shown in this paper that the relative acceleration between neighbouring particles in the field of an isolated mass is finite at the apparent singular surface $r = 2m$. The analysis is made both in Schwarzschild and Eddington–Finkelstein coordinate systems by using the equation of geodesic deviation.

1. *Introduction*

The role of singularity in the study of space-time structure has been of immense interest though the definition of the singularity itself has not been generally accepted. In the study of the gravitational field of an isolated mass, the Schwarzschild surface $r = 2m$ was once thought to be singular until the diagrammatic representation of its analytic extension was made (Fronsdal, 1959; Kruskal, 1960). So far, the arguments put forward for concluding that the Schwarzschild surface is non-singular in the space-time manifold have been mainly on the basis of the analysis of the light-cone along a radial geodesic (Finkelstein, 1959; Misner, 1968). This has eventually led to the conclusion that the surface forms only an absolute event horizon (Penrose, 1969).

In this paper we have shown that it is possible to arrive at the conclusion that the surface $r = 2m$ is non-singular on purely physical grounds by considering the relative acceleration of freely falling particles studied through the equations of geodesic deviation. In Section 2 we give the basic mathematical vocabulary of the equations of geodesic deviation and the method of obtaining relative acceleration. In Section 3 the tetrads have been constructed to which the curvature tensor R_{hijk} is referred and the components of relative acceleration are obtained both in Schwarzschild coordinates and Eddington–Finkelstein coordinates. In conclusion, an application of this analysis is mentioned by considering an atom on the surface of a collapsing sphere as a system of two freely falling particles in the gravitational field.

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2. *Equations of Geodesic Deviation*

The geometrical significance of geodesic deviation has been studied by Synge and Levi-Civita. Synge (1964) has given an elegant method of obtaining the equations of geodesic deviation in the form†

$$\frac{\partial^2 \eta_i}{\partial s^2} + R^i_{jkl}\,\lambda^j\,\eta^k\,\lambda^l = 0 \tag{2.1}$$

where η_i denotes the infinitesimal displacement from a point $P(u,v)$ on a geodesic A to the point $Q(u,v+\delta v)$ on B, a neighbouring geodesic, R^i_{jkl} the curvature tensor, $\lambda^i = dx^i/ds$ the unit tangent vector to the geodesic world-line A of one of the particles and s the proper time along A. In order to get the physical interpretation of (2.1), one could compare it with the analogous equations of Newtonian physics. Pirani (1957) has done this by referring (2.1) to a tetrad of which λ^j is the time-like member (the four-velocity of the particle with world-line A) and the space-like triad $\lambda_\alpha{}^i$ chosen such that it propagates parallel along the chosen geodesic A. Resolving η^i, the displacement vector along the triad $\lambda_\alpha{}^i$, we get

$$\eta^i = X_\alpha\,\lambda_\alpha{}^i \tag{2.2}$$

where X_α are three scalar point function. Using (2.2) in (2.1) we get the invariant form of the deviation equations

$$\frac{d^2 X_\alpha}{ds^2} + K_{\alpha\beta}\,X_\beta = 0 \tag{2.3}$$

where

$$K_{\alpha\beta} = R_{hijk}\,\lambda^h\,\lambda_\alpha{}^i\,\lambda^j\,\lambda_\beta{}^k \tag{2.4}$$

represents the relative acceleration between the two particles under consideration.

In order to evaluate the acceleration components, we consider the metric of the space-time in which the particles are freely falling and solve the equation of geodesics

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0 \tag{2.5}$$

for any one of them to get the components of four-velocity dx^i/ds $(=\lambda^i)$.

Next we have to construct the orthonormal triad $\lambda^i_{(\alpha)}$ such that

$$g_{ij}\,\lambda^i\,\lambda_\alpha{}^j = 0, \qquad g_{ij}\,\lambda^i_{(\alpha)}\,\lambda^j_{(\alpha)} = -1 \tag{2.6}$$

Having thus obtained the components of the tetrad $\lambda_k{}^i = (\lambda_\alpha{}^i, \lambda^i)$, we can compute the acceleration components $K_{\alpha\beta}$.

† We use the notation that Latin indices take values 1, 2, 3, 4 (x^4 = time coordinate) whereas the Greek indices take the values 1, 2, 3.

3. *Acceleration Components*

A. Schwarzschild Coordinates

The field of an isolated mass particle is given by the well-known space-time metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) + \left(1 - \frac{2m}{r}\right) dt^2 \quad (3.1)$$

In this field the equations of motion (2.5) for a freely falling particle along the radial direction ($\theta = \phi =$ const) are given by

$$\left.\begin{aligned} \frac{d^2 r}{ds^2} - \frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1}\left(\frac{dr}{ds}\right)^2 + \frac{m}{r^2}\left(1 - \frac{2m}{r}\right)\left(\frac{dt}{ds}\right)^2 &= 0 \\ \frac{d^2 t}{ds^2} + \frac{2m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1} &= 0 \end{aligned}\right\} \quad (3.2)$$

Solving (3.2), using (3.1), we get the four-velocity

$$\lambda^i = \left\{\left(\frac{2m}{r} - \frac{2m}{r_0}\right)^{1/2}, 0, 0, \left(1 - \frac{2m}{r_0}\right)^{1/2}\left(1 - \frac{2m}{r}\right)^{-1}\right\} \quad (3.3)$$

where at $r = r_0$ the particle is assumed to be at rest.

We now obtain the space-like triad $\lambda^i_{(\alpha)}$ after using (3.1) and (3.3) in (2.6) to be

$$\begin{aligned} \lambda_1{}^i &= \left\{\left(1 - \frac{2m}{r_0}\right)^{1/2}, 0, 0, \left(\frac{2m}{r} - \frac{2m}{r_0}\right)^{1/2}\left(1 - \frac{2m}{r}\right)^{-1}\right\} \\ \lambda_2{}^i &= \left\{0, \frac{1}{r}, 0, 0\right\} \\ \lambda_3{}^i &= \left\{0, 0, \frac{1}{r\sin\theta}, 0\right\} \end{aligned} \quad (3.4)$$

Thus the tetrad $\lambda_k{}^i = (\lambda_\alpha{}^i, \lambda^i)$ is determined.

For the space-time (3.1) the non-vanishing independent components of R_{hijk} are given by

$$\left.\begin{aligned} R_{1212} &= R_{1313}/\sin^2\theta = \frac{m}{r}\left(1 - \frac{2m}{r}\right)^{-1} \\ R_{2424} &= R_{3434}/\sin^2\theta = -\frac{m}{r}\left(1 - \frac{2m}{r}\right) \\ R_{1414} &= \frac{2m}{r^3} \\ \frac{R_{2323}}{\sin^2\theta} &= -2mr \end{aligned}\right\} \quad (3.5)$$

Using (3.3), (3.4) and (3.5) in (2.4), we find that

$$K_{11} = \frac{2m}{r^3}, \qquad K_{22} = K_{33} = -\frac{m}{r^3}$$
$$K_{\alpha\beta} = 0, \qquad \alpha \neq \beta \tag{3.6}$$

B. Eddington–Finkelstein Coordinates

The space-time metric is now given by (Eddington, 1924; Finkelstein, 1959)

$$ds^2 = +\left(1 - \frac{2m}{r}\right) du^2 - 2\,dr\,du - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{3.7}$$

The equations of motion of one of the particles are now given by

$$\frac{d^2 r}{ds^2} - \frac{2m}{r^2}\frac{dr}{ds}\frac{du}{ds} + \frac{m}{r^2}\left(1 - \frac{2m}{r}\right)\left(\frac{du}{ds}\right)^2 = 0$$
$$\frac{d^2 u}{ds^2} + \frac{m}{r^2}\left(\frac{du}{ds}\right)^2 = 0 \tag{3.8}$$

which on integration give

$$\frac{dr}{ds} = \left(\frac{2m}{r} - \frac{2m}{r_0}\right)^{1/2}$$
$$\frac{du}{ds} = \left[\left(\frac{2m}{r} - \frac{2m}{r_0}\right)^{1/2} \pm \left(1 - \frac{2m}{r_0}\right)^{1/2}\right]\left(1 - \frac{2m}{r}\right)^{-1} \tag{3.9}$$

In the last of (3.9), by choosing the negative sign we can avoid the infinity at $r = 2m$.

The space-like triad $\lambda_\alpha{}^i$ is now chosen to be

$$\lambda_{(1)}^i = \left\{\left(1 - \frac{2m}{r_0}\right)^{1/2}, 0, 0, \left(1 - \frac{2m}{r}\right)^{-1}\left[\left(1 - \frac{2m}{r_0}\right)^{1/2} - \left(\frac{2m}{r} - \frac{2m}{r_0}\right)^{1/2}\right]\right\}$$
$$\lambda_{(2)}^i = \left\{0, \frac{1}{r}, 0, 0\right\} \tag{3.10}$$
$$\lambda_3{}^i = \left\{0, 0, \frac{1}{r\sin\theta}, 0\right\}$$

such that the tetrad $\lambda_k{}^i = (\lambda_\alpha{}^i, \lambda^i)$, where

$$\lambda^i = \left\{\left(\frac{2m}{r} - \frac{2m}{r_0}\right)^{1/2}, 0, 0, \left[\left(\frac{2m}{r} - \frac{2m}{r_0}\right)^{1/2} - \left(1 - \frac{2m}{r_0}\right)^{1/2}\right]\left(1 - \frac{2m}{r}\right)^{-1}\right\} \tag{3.11}$$

satisfies equations (2.6).

For (3.7) the non-vanishing independent components of R_{hijk} are given by

$$\left.\begin{aligned} R_{1414} &= \frac{2m}{r^3} \\ R_{2323}/\sin^2\theta &= -2mr \\ R_{2424} = R_{3434}/\sin^2\theta &= -\frac{m}{r}\left(1-\frac{2m}{r}\right) \\ R_{1224} = R_{1334}/\sin^2\theta &= -\frac{m}{r} \end{aligned}\right\} \tag{3.12}$$

Now (3.10)–(3.12), when used in (2.4), give again the acceleration components to be

$$\begin{aligned} &K_{11} = \frac{2m}{r^3}, \qquad K_{22} = K_{33} = -\frac{m}{r^3} \\ &K_{\alpha\beta} = 0 \qquad \text{for } \alpha \neq \beta \end{aligned} \tag{3.13}$$

Thus we find that the acceleration components remain the same in both systems of coordinates, as is to be expected, for the physical components of the curvature tensor do not depend on the choice of the coordinate system. It is interesting to see that at $r = 2m$, though g_{44} vanishes in both the systems, the fourth component of the four-velocity dx^i/ds is infinite in S coordinates whereas it is finite in E–F coordinates.

After this paper was completed we saw a paper by J. D. Finley III (1971) wherein the author has arrived at a similar result regarding the finiteness of the relative acceleration at $r = 2m$ by using a more complicated tetrad.

4. *Conclusion*

As can be seen from the values of the components of relative acceleration between freely falling test particles (in radial collapse) at $r = 2m$ nothing disastrous happens. On the other hand, as $r \to 0$ these accelerations tend to infinity. Since the signs are different for radial and transverse components, the particles get torn apart by the tidal effects as they approach the origin.

The analysis made above can be very significant while we consider the motion of an atom (like a hydrogen atom) sitting on the surface of a freely collapsing spherical body. If we now consider the nucleus and the electron independently as two particles on different but infinitesimally close geodesics, we find that because of the curvature there is a relative acceleration between them of strength as found above. This acceleration keeps on increasing proportional to $1/r^3$ as the body collapses and the system of these two particles (atom) is subjected to tidal forces. It would be interesting to find out for what value of r (non-zero) the tidal accelerations take over the binding force of the atom causing the atom to split up.

Acknowledgements

The author wishes to thank Prof. Abdus Salam, the International Atomic Energy Agency and UNESCO for hospitality at the International Centre for Theoretical Physics, Trieste.

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